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The $\mathcal{PT}$-symmetric Rosen–Morse II potential: effects of the asymptotically non-vanishing imaginary potential component

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Abstract

Bound and scattering solutions of the $\mathcal{PT}$-symmetric Rosen–Morse II potential are investigated. The energy eigenvalues and the corresponding wavefunctions are written in a closed analytic form, and it is shown that this potential always supports at least one bound state. It is found that with increasing non-Hermiticity the real bound-state energy spectrum does not turn into complex conjugate pairs, i.e. the spontaneous breakdown of $\mathcal{PT}$ symmetry does not occur, rather the energy eigenvalues remain real and shift to positive values. Closed expression is found for the pseudo-norm of the bound states, and its sign is found to follow the $(-1)^n$ rule. Similarly to the known scattering examples, the reflection coefficients exhibit a handedness effect, while the transmission coefficient picks up a complex phase factor when the direction of the incoming wave is reversed. It is argued that the unusual findings might be caused by the asymptotically non-vanishing, though finite imaginary potential component. Comparison with the real Rosen–Morse II potential is also made.

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1. Introduction

The investigation of non-Hermitian quantum mechanical systems has gone through a renaissance in the past decade after the introduction of $\mathcal{PT}$-symmetric quantum mechanics [1] (for a review see, e.g., [2]). It turned out that manifestly non-Hermitian quantum Hamiltonians can possess partly or fully real energy spectra and other features (e.g., the conservation of the norm) that have been usually associated with Hermitian systems. These studies strengthened anew the importance of the relation between the Hermiticity of a Hamiltonian and the choice of the inner product and the space it is defined on. Non-Hermitian models making use of modified metric operators have been known previously under different names, e.g., pseudo- [3], quasi- [4] and crypto-Hermiticity [5], and these theories have enjoyed renewed interest recently.

$\mathcal{PT}$-symmetric quantum mechanics was also identified as a special case of pseudo-Hermiticity [6]. In the case of the one-dimensional Schrödinger equation $\mathcal{PT}$ symmetry, i.e., the invariance of the Hamiltonian under the joint action of the $\mathcal{P}$ space and $\mathcal{T}$ time inversion prescribes the $V^*(-x) = V(x)$ relation for the potential. The first examples for $\mathcal{PT}$-symmetric potentials were studied numerically and included the archetypical example $V(x) = x^2(\mathrm{i}x)^\epsilon$ [1]. This potential possesses the real and positive energy spectrum for $\epsilon \geqslant 0$, while shifting it below this value the energy eigenvalues gradually merge pairwise and reemerge as complex conjugate pairs. This mechanism was interpreted as the spontaneous breakdown of $\mathcal{PT}$ symmetry, as the eigenfunctions ceased to be the eigenfunctions of the $\mathcal{PT}$ operator then.

Soon after the introduction of $\mathcal{PT}$-symmetric quantum mechanics, the investigation into the $\mathcal{PT}$-symmetric versions of exactly solvable potentials was started. The importance of these studies lies in the fact that the exact analytic formulation of such systems might allow deeper insight into the mechanisms underlying $\mathcal{PT}$ symmetry including also its spontaneous breakdown. The $\mathcal{PT}$-symmetric version of a number of solvable potentials has been described including the simplest shape-invariant class [7–14], the more general Natanzon class [15, 16] and even potentials beyond it [17]. Investigations focused on features characteristic of $\mathcal{PT}$-symmetry, such as the formulation of conditions for real and complex eigenvalues [12–14], the mechanism of the spontaneous breakdown of $\mathcal{PT}$ symmetry [18, 19], the connection of algebraic [20–22] and supersymmetric [12, 23] structures with $\mathcal{PT}$ symmetry, the analysis of the pseudo-norm [19, 24, 25] and the exact construction of the $\mathcal{C}$ operator [26].

Scattering aspects of $\mathcal{PT}$-symmetric potentials have also been studied in the exact analytical formulation. For this, asymptotically vanishing potentials were selected [21, 22, 27, 28] and the implications of $\mathcal{PT}$ symmetry have been analyzed. Besides the manifest non-unitarity of the scattering results, a remarkable finding was the handedness of the potentials [27] meaning that the reflection coefficient depended on whether the wave arrived from the absorptive or the emissive regime of the odd imaginary potential component.

Here we study the $\mathcal{PT}$-symmetric Rosen–Morse II potential, defined on the real x-axis, and investigate both its bound and scattering states. The novel feature with respect to other $\mathcal{PT}$-symmetric potentials is that although its real component vanishes asymptotically, its imaginary component does not, although it remains finite. We aim at exploring the consequences of these unusual features of the $\mathcal{PT}$-symmetric Rosen–Morse II potential, rather than proposing the application of this potential to specific physical problems. It is worth mentioning though that this potential could describe a system in which an emissive and an absorptive domain are separated by a finite real, short-range potential well. Its limiting case would be a real Dirac δ-potential at $x = 0$ that separates the two domains of an odd imaginary step potential.

In section 2, we establish the notation in a general form that includes both the $\mathcal{PT}$-symmetric and real Rosen–Morse II potential. In section 3, we specify it to the $\mathcal{PT}$-symmetric case and analyze the bound and scattering states in two respective subsections. For the sake of completeness, we also discuss the real Rosen–Morse II potential in section 4, while the results are discussed in section 5.

2. The general formalism

The Rosen–Morse II (in short: RM II) potential belongs to the same family of exactly solvable potentials as its trigonometric counterpart, the Rosen–Morse I (RM I) and also the Eckart potential. This family has been identified as the PII class within a transformation method [29] which links the bound-state eigenfunctions of the corresponding Schrödinger equations with the Jacobi polynomials. The same family is denoted as Type E in terms of the factorization method [30].

Following the notation of [13, 14], the fundamental solutions $\psi^{(\pm)}(x)$ of the Schrödinger equation

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + (E - V(x))\psi = 0 \tag{1}$$

with the Rosen–Morse II potential

$$V(x) = -\frac{s(s+1)}{\cosh^2(x)} + 2\Lambda\tanh(x) \tag{2}$$

can be written in terms of the hypergeometric function [31] $F(a, b; c; z)$ in the forms

$$\psi^{(+)}(x) \sim (1-\tanh x)^{\alpha/2}(1+\tanh x)^{\beta/2} F\left(-s + \frac{\alpha+\beta}{2}, s+1+\frac{\alpha+\beta}{2}; 1+\alpha; \frac{1-\tanh x}{2}\right) \tag{3}$$

and

$$\psi^{(-)}(x) \sim (1-\tanh x)^{-\alpha/2}(1+\tanh x)^{\beta/2} \times F\left(-s + \frac{\beta-\alpha}{2}, s+1+\frac{\beta-\alpha}{2}; 1-\alpha; \frac{1-\tanh x}{2}\right), \tag{4}$$

where α and β are two new parameters, specified in terms of E and Λ by equations [29]

$$\left(\frac{\alpha+\beta}{2}\right)^2 + \left(\frac{\beta-\alpha}{2}\right)^2 + E = 0, \qquad \frac{\alpha+\beta}{2}\frac{\beta-\alpha}{2} + \Lambda = 0. \tag{5}$$

Note that (3) and (4) are connected by the transformation $\alpha \leftrightarrow -\alpha$. Concerning the two parameters involved in the potential (2), for the moment we assume that s is real and positive, while Λ can be either real or pure imaginary. In the latter case, $\Lambda = \mathrm{i}\lambda$, $\lambda \in R$ the complex Rosen–Morse II potential (2) is $\mathcal{PT}$-symmetric [13].

Note that the $s \leftrightarrow -s-1$ transformation leaves both the potential (2) and the solutions (3) and (4) invariant, the $s < -1$ domain is also covered automatically. We shall comment on the remaining $s \in [-1, 0]$ domain separately, which corresponds to the coupling constant $s(s+1) \in \left[-\frac{1}{4}, 0\right]$ in the first term of (2). We shall also see that certain complex values of s are also admissible in the case of the $\mathcal{PT}$-symmetric Rosen–Morse II potential.

Bearing in mind the asymptotic properties of the hypergeometric function [31] we easily obtain

$$\psi^{(+)}(x \to -\infty) = 2^{(\beta+\alpha)/2}\left[\frac{\Gamma(\alpha+1)\Gamma(-\beta)\exp(\beta x)}{\Gamma\left(-s+\frac{\alpha-\beta}{2}\right)\Gamma\left(s+1+\frac{\alpha-\beta}{2}\right)} + \frac{\Gamma(\alpha+1)\Gamma(\beta)\exp(-\beta x)}{\Gamma\left(-s+\frac{\alpha+\beta}{2}\right)\Gamma\left(s+1+\frac{\alpha+\beta}{2}\right)}\right] \tag{6}$$

$$\psi^{(-)}(x \to -\infty) = 2^{(\beta-\alpha)/2}\left[\frac{\Gamma(-\alpha+1)\Gamma(-\beta)\exp(\beta x)}{\Gamma\left(-s-\frac{\alpha+\beta}{2}\right)\Gamma\left(s+1-\frac{\alpha+\beta}{2}\right)} + \frac{\Gamma(-\alpha+1)\Gamma(\beta)\exp(-\beta x)}{\Gamma\left(-s+\frac{\beta-\alpha}{2}\right)\Gamma\left(s+1+\frac{\beta-\alpha}{2}\right)}\right] \tag{7}$$

$$\psi^{(+)}(x \to +\infty) = 2^{(\beta+\alpha)/2} \exp(-\alpha x) \tag{8}$$

$$\psi^{(-)}(x \to +\infty) = 2^{(\beta-\alpha)/2} \exp(\alpha x). \tag{9}$$

These equations are useful when dealing with the scattering solutions of the Rosen–Morse II potential. Note that the two terms inside the parentheses in (6) and (7) are connected by the $\beta \leftrightarrow -\beta$ relation. This is due to a basic transformation property of the hypergeometric functions seen, e.g., in equation (15.3.3) of [31], which takes equations (3) and (4) into equivalent forms in which $-\beta$ appears instead of β everywhere (and also introduces a constant factor 2^{β}). This means that similarly to the sign of α, the sign of β does not influence the results either. We will use this freedom to chose the signs in a way that results in formulae similar to those obtained for the real Rosen–Morse II potential previously (see, e.g., [13, 29] and references therein).

Equations (6)–(9) express the asymptotic behavior of the solutions in terms of the α and β parameters, which in turn are directly related to the asymptotic wave numbers in the $x \to \pm\infty$ limit. This is because it can be shown that

$$E - V(-\infty) = E + 2\Lambda = -\beta^2 \tag{10}$$

$$E - V(+\infty) = E - 2\Lambda = -\alpha^2, \tag{11}$$

so the wave numbers can be expressed as

$$k = \mathrm{i}\beta, \qquad k' = \mathrm{i}\alpha \tag{12}$$

in the two limits. Obviously, since the odd component of the potential does not vanish asymptotically, $k \neq k'$ will hold in general.

3. The complex $\mathcal{PT}$-symmetric Rosen–Morse II potential

When $\Lambda = \mathrm{i}\lambda$ with $\lambda \in R$, the Rosen–Morse II potential is complex and $\mathcal{PT}$-symmetric. The aim of this section is to investigate the bound and scattering sates in the potential

$$V(x) = -\frac{s(s+1)}{\cosh^2(x)} + 2\,\mathrm{i}\lambda\tanh(x). \tag{13}$$

3.1. Bound states

Equations (5) now result in

$$E = -\left(\frac{\alpha+\beta}{2}\right)^2 + \left(\frac{2\lambda}{\beta+\alpha}\right)^2. \tag{14}$$

The bound states correspond to the boundary conditions

$$\lim_{x\to\pm\infty} \psi(x) = 0. \tag{15}$$

Equations (8), (9) and (15) show that the bound-state solutions are obtained either for $\mathrm{Re}(\alpha) > 0$ or for $\mathrm{Re}(\alpha) < 0$, and are described by the solutions $\psi^{(+)}(x)$ or $\psi^{(-)}(x)$, respectively. The two cases are mutually excluding. For this reason, we restrict our attention to the solution $\psi^{(+)}(x)$ in (3).

The regularity of solution (3) requires $\mathrm{Re}(\beta) > 0$ in addition to $\mathrm{Re}(\alpha) > 0$. Accordingly, the function (6) satisfies the asymptotic condition (15) as $x \to -\infty$ when, e.g.,

$$\frac{\alpha+\beta}{2} - s = -n, \qquad n = 0, 1, \ldots, n_{\max}. \tag{16}$$

Thus, the energies of the bound states are real

$$E_n = -(s-n)^2 + \frac{\lambda^2}{(s-n)^2}, \qquad n = 0, 1, \ldots, n_{\max}. \tag{17}$$

The corresponding eigenfunctions (3) can be expressed in terms of the Jacobi polynomials as [13, 29]

$$\psi_n(x) = C_n(1-\tanh x)^{\frac{\alpha}{2}}(1+\tanh x)^{\frac{\beta}{2}} P_n^{(\alpha,\beta)}(\tanh x), \tag{18}$$

where C_n is a normalization constant and

$$\alpha_n = s - n + \frac{\mathrm{i}\lambda}{s-n}, \qquad \beta_n = s - n - \frac{\mathrm{i}\lambda}{s-n}. \tag{19}$$

According to the regularity conditions $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\beta) > 0$, equations (19) imply that the number of bound states is always finite,

$$n_{\max} < s. \tag{20}$$

Before going on it is worth mentioning that the $\mathcal{PT}$ symmetry, in principle, allows complex values of s too as $s = -\frac{1}{2} + \mathrm{i}\sigma$ [14]. In this case, the coupling coefficient of the $\cosh^{-2}(x)$ term becomes definitely positive, $\frac{1}{4} + \sigma^2$, so the real component of the potential (2) turns into a finite barrier. Furthermore, a straightforward calculation shows that the two conditions of normalizability $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\beta) > 0$ cannot be fulfilled at the same time, so this implies that no regular solutions can exist in this case. The same holds in the $s \in [-1, 0]$ domain too, which was not included in the analysis before.

Let us now calculate the pseudo-norm of the functions (18). First we note that in order to make these functions the eigenfunctions of the $\mathcal{PT}$ operator with unit eigenvalue $\mathcal{PT}\psi_n(x) = \psi_n(x)$, the normalization constant C_n has to be chosen as

$$C_n = \mathrm{i}^n c_n, \qquad c_n \in R. \tag{21}$$

This follows from the relation $\alpha_n = \beta_n^*$ and the transformation properties of the Jacobi polynomials with respect to changing their argument as $z \to -z$ [31]. Making use of the defining formula of the Jacobi polynomials (e.g. equation (22.3.1) in [31]) one obtains

$$\begin{aligned} I_{nm} \equiv \langle\psi_n|\mathcal{P}|\psi_m\rangle &= \int_{-\infty}^{\infty} \psi_n(x)[\psi_m(-x)]^*\,\mathrm{d}x = \int_{-\infty}^{\infty} \psi_n(x)\psi_m(x)\,\mathrm{d}x \\ &= C_n C_m (-1)^{n+m} 2^{(\alpha_n+\beta_n+\alpha_m+\beta_m)/2-1}\left[\Gamma\left(\frac{\alpha_n+\beta_n+\alpha_m+\beta_m}{2}+n+m\right)\right]^{-1} \\ &\quad \times \sum_{k=0}^{n}(-1)^k \binom{\alpha_n+n}{k}\binom{\beta_n+n}{n-k}\sum_{j=0}^{m}(-1)^j\binom{\alpha_m+m}{j}\binom{\beta_m+m}{m-j} \\ &\quad \times \Gamma\left(\frac{\alpha_n+\alpha_m}{2}+n+m-k-j\right)\Gamma\left(\frac{\beta_n+\beta_m}{2}+k+j\right). \end{aligned} \tag{22}$$

Here the integration was performed using equation (2.5.30.3) in [32]. The second equation in (22) follows from the $\mathcal{PT}\psi_m(x) = \psi_m(x)$ requirement. It is notable that the structure of (22) is similar to the corresponding formula obtained for the $\mathcal{PT}$-symmetric (trigonometric) Rosen–Morse I potential [25].

It is not obvious to prove the $\mathcal{PT}$-orthogonality of different states from equation (22); however, this can be done indirectly using the standard formula obtained from the respective Schrödinger equations and integrating by part:

$$(E_n - [E_m]^*)\int_{-\infty}^{\infty}\psi_n(x)[\psi_m(-x)]^*\,\mathrm{d}x = 0. \tag{23}$$

Here we made use of the result that the wavefunctions (18) and their derivatives vanish asymptotically.

The normalization constant can be evaluated from (22) after taking $n = m$:

$$I_{nn} = C_n^2 2^{\alpha_n+\beta_n-1} \frac{\Gamma(\alpha_n+n+1)\Gamma(\beta_n+n+1)}{n!\Gamma(\alpha_n+\beta_n+n+1)} \left(\frac{1}{\alpha_n}+\frac{1}{\beta_n}\right) \tag{24}$$

$$= (-1)^n c_n^2 2^{2s-2n} \frac{|\Gamma(s+1+\mathrm{i}\lambda/(s-n))|^2}{n!\Gamma(2s-n+1)} \frac{s-n}{(s-n)^2+\lambda^2/(s-n)^2}. \tag{25}$$

In order to evaluate the double sum in (22) we applied equations (4.2.2.3) and (4.2.2.43) from [32]. It is clear that due to the $s > n$ regularity condition all the components of I_{nn} are positive, except for the $(-1)^n$ term, so the sign of the pseudo-norm strictly follows the $(-1)^n$ rule. This is similar to that observed for the $\mathcal{PT}$-symmetric Scarf I [24] and Rosen–Morse I [25] potentials as well as for other $\mathcal{PT}$-symmetric potentials with an infinite number of bound states [33, 34], but is different from the case of the $\mathcal{PT}$-symmetric Scarf II potential [19], which also has a finite number of bound states. Eventually, the normalization constant appearing in the wavefunctions of the $\mathcal{PT}$-symmetric Rosen–Morse II potential is

$$C_n = \frac{\mathrm{i}^n 2^{n-s}}{|\Gamma(s+1+\mathrm{i}\lambda/(s-n))|} \left(\frac{n!\Gamma(2s-n+1)[(s-n)^2+\lambda^2/(s-n)^2]}{s-n}\right)^{1/2}. \tag{26}$$

3.2. *Scattering states*

Since $\Lambda = \mathrm{i}\lambda$ is purely imaginary, the non-vanishing odd component of the potential will be purely imaginary in this case. In particular, for $\lambda > 0$ the imaginary potential component will be absorptive for $x < 0$ and emissive for $x > 0$, respectively. For this reason the asymptotic wave numbers (12) obtained from (10) and (11) will be complex.

Considering the parametrization

$$k = k_R + \mathrm{i}k_I, \qquad k' = k'_R + \mathrm{i}k'_I \tag{27}$$

and assuming that E is real, the relations

$$E = k_R^2 - k_I^2 = k_R'^2 - k_I'^2 \tag{28}$$

$$\lambda = k_R k_I = -k'_R k'_I \tag{29}$$

follow. Direct calculations show that the solution of this set of equations is $k' = \pm k^*$. In the case of complex wave numbers equations (10) and (11) can be used to determine the correct relation of k and k': for $\lambda > 0$ and $E > 0, k^2 = E + 2\mathrm{i}\lambda = [(k')^2]^*$ implies that $0 < \mathrm{Arg}(k) < \pi/4$ and $3\pi/4 < \mathrm{Arg}(k') < \pi$, i.e., the $k' = -k^*$ relation must hold. This means $k_R = -k'_R > 0$ and $k_I = k'_I > 0$. (For $\lambda < 0$ the roles of k and k' are exchanged.)

Considering equation (12), the asymptotic expansions (6)–(9) can be written as

$$\psi^{(+)}(x \to -\infty) \sim a_-^{(+)} \mathrm{e}^{\mathrm{i}k_R x} \mathrm{e}^{-k_I x} + b_-^{(+)} \mathrm{e}^{-\mathrm{i}k_R x} \mathrm{e}^{k_I x} \tag{30}$$

$$\psi^{(+)}(x \to +\infty) \sim a_+^{(+)} \mathrm{e}^{\mathrm{i}k'_R x} \mathrm{e}^{-k'_I x} + b_+^{(+)} \mathrm{e}^{-\mathrm{i}k'_R x} \mathrm{e}^{k'_I x} \tag{31}$$

$$\psi^{(-)}(x \to -\infty) \sim a_-^{(-)} \mathrm{e}^{\mathrm{i}k_R x} \mathrm{e}^{-k_I x} + b_-^{(-)} \mathrm{e}^{-\mathrm{i}k_R x} \mathrm{e}^{k_I x} \tag{32}$$

$$\psi^{(-)}(x \to +\infty) \sim a_+^{(-)} \mathrm{e}^{\mathrm{i}k'_R x} \mathrm{e}^{-k'_I x} + b_+^{(-)} \mathrm{e}^{-\mathrm{i}k'_R x} \mathrm{e}^{k'_I x}, \tag{33}$$

where

$$a_-^{(+)} = \frac{2^{(\beta+\alpha)/2}\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma\left(-s+\frac{\alpha+\beta}{2}\right)\Gamma\left(s+1+\frac{\alpha+\beta}{2}\right)} \qquad b_-^{(+)} = \frac{2^{(\beta+\alpha)/2}\Gamma(\alpha+1)\Gamma(-\beta)}{\Gamma\left(-s+\frac{\alpha-\beta}{2}\right)\Gamma\left(s+1+\frac{\alpha-\beta}{2}\right)}, \tag{34}$$

$$a_-^{(-)} = \frac{2^{(\beta-\alpha)/2}\Gamma(-\alpha+1)\Gamma(\beta)}{\Gamma\left(-s+\frac{\beta-\alpha}{2}\right)\Gamma\left(s+1+\frac{\beta-\alpha}{2}\right)} \qquad b_-^{(-)} = \frac{2^{(\beta-\alpha)/2}\Gamma(-\alpha+1)\Gamma(-\beta)}{\Gamma\left(-s-\frac{\alpha+\beta}{2}\right)\Gamma\left(s+1-\frac{\alpha+\beta}{2}\right)}, \tag{35}$$

$$a_+^{(+)} = 2^{(\beta+\alpha)/2} \qquad b_+^{(+)} = 0, \tag{36}$$

$$a_+^{(-)} = 0 \qquad b_+^{(+)} = 2^{(\beta-\alpha)/2}. \tag{37}$$

The transmission and reflection coefficients can be calculated in the standard way by combining the asymptotic expansions of the two solutions. For example, for $\lambda > 0$ (i.e., $k_R > 0$ and $k'_R < 0$) an incoming wave from the left and the reflected wave are represented by the first and second terms of (30) and (32), respectively, while the transmitted wave comes from the second terms of (31) and (33). At the same time, the linear combination coefficients have to be chosen such that the terms with $\mathrm{e}^{\mathrm{i}k'_R x}$ should vanish. The transmission and reflection coefficients for a wave coming from the left are

$$T_{L\to R}(k,k') = \frac{\Gamma(-s-\mathrm{i}k/2+\mathrm{i}k'/2)\Gamma(s+1-\mathrm{i}k/2+\mathrm{i}k'/2)}{\Gamma(1+\mathrm{i}k')\Gamma(-\mathrm{i}k)} \tag{38}$$

$$R_{L\to R}(k,k') = T_{L\to R}(k,k')\frac{\Gamma(1+\mathrm{i}k')\Gamma(\mathrm{i}k)}{\Gamma(-s+\mathrm{i}(k+k')/2)\Gamma(s+1+\mathrm{i}(k+k')/2)}. \tag{39}$$

The bound-state solutions are obtained from the poles of the transmission coefficient (38), i.e., by setting the argument of the first gamma functions in the numerator to a non-positive integer. With this and equations (5) the relation (16) is reproduced, while considering the second gamma function in the same manner simply corresponds to the $s \leftrightarrow -s-1$ replacement. It is notable that the λ coupling coefficient of the odd potential component appears in $T(k,k')$ and $R(k,k')$ only implicitly via k and k'.

The procedure can be repeated for an incoming wave from the right. The importance of the odd imaginary component of asymptotically vanishing $\mathcal{PT}$-symmetric potentials has already been pointed out: in [27] this was discussed in connection with the handedness of the problem. This phenomenon also seems to be important for potentials of the Rosen–Morse II kind that possess an attractive and asymptotically vanishing even real well in addition to the odd imaginary component. One important novelty is that in this case the imaginary potential component does not vanish asymptotically. The transmission and reflection coefficients for an incoming wave from the right are

$$T_{R\to L}(k,k') = -\frac{k'}{k}T_{L\to R}(k,k') \tag{40}$$

$$R_{R\to L}(k,k') = R_{L\to R}(k,k')\frac{\Gamma(-\mathrm{i}k')}{\Gamma(\mathrm{i}k')}\frac{\Gamma(-\mathrm{i}k)}{\Gamma(\mathrm{i}k)}\frac{\Gamma(-s+\mathrm{i}(k+k')/2)}{\Gamma(-s-\mathrm{i}(k'+k)/2)}\frac{\Gamma(s+1+\mathrm{i}(k+k')/2)}{\Gamma(s+1-\mathrm{i}(k'+k)/2)}. \tag{41}$$

Equations (40) and (41) indicate that similar to asymptotically vanishing $\mathcal{PT}$-symmetric potentials [27], the reflection coefficient exhibits the handedness effect, while changing the direction of the incoming wave modifies the transmission coefficient by complex phase factor k^*/k.

As discussed previously, results for $\lambda < 0$ can be obtained by replacing the roles of k and k'.

4. The real Rosen–Morse II potential

For the sake of completeness, we summarize the corresponding results for the real Rosen–Morse II potential with $\Lambda = \lambda$, $\lambda \in R$ [35].

All the equations of section 2 are valid for both the real and complex Rosen–Morse II potentials. When $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\beta) > 0$, the regular solution is $\psi^{(+)}(x)$ also for the real Rosen–Morse II potential (2). The requirement that the asymptotic form (6) of $\psi^{(+)}(x)$ for $x \to -\infty$ satisfy the corresponding condition (15) leads also in this case to the same bound-state condition (16) as in the case of the complex Rosen–Morse II potential. Accordingly, the bound-state energies are obtained now from equations (5) as

$$E_n = -(s-n)^2 - \frac{\lambda^2}{(s-n)^2}, \qquad n = 0, 1, \ldots, n_{\max}. \tag{42}$$

Although the regularity conditions are formally the same in both the real and complex potentials, the actual content and implications of these conditions are basically different in the two cases. Indeed in the present case, the place of equations (15) is taken over by

$$\alpha_n = s - n + \frac{\lambda}{s-n}, \qquad \beta_n = s - n - \frac{\lambda}{s-n}. \tag{43}$$

The bound-state eigenfunctions are obtained now from the same equation (18), but with α_n and β_n given by equations (43) and with different values of the normalization constants C_n (see below). However, in contrast to equation (20), the largest value $n_{\max}$ of the principal quantum number is limited in this case not by s alone, but by a smaller quantity, namely

$$n_{\max} < s - |\lambda|^{1/2}. \tag{44}$$

The different choice of Λ is also reflected by the wavefunctions (18) in that they can now be written in a purely real form. The Hermitian inner product of these functions can be calculated in a way similar to that described previously, and the results will also be rather similar formally:

$$\begin{aligned} J_{nm} \equiv \langle \psi_n | \psi_m \rangle &= \int_{-\infty}^{\infty} \psi_n(x)[\psi_m(x)]^* \,\mathrm{d}x \\ &= C_n C_m^* (-1)^{n+m} 2^{(\alpha_n+\beta_n+\alpha_m+\beta_m)/2-1} \left[\Gamma\left(\frac{\alpha_n+\beta_n+\alpha_m+\beta_m}{2} + n + m \right) \right]^{-1} \\ &\quad \times \sum_{k=0}^{n} (-1)^k \binom{\alpha_n+n}{k} \binom{\beta_n+n}{n-k} \sum_{j=0}^{m} (-1)^j \binom{\alpha_m+m}{j} \binom{\beta_m+m}{m-j} \\ &\quad \times \Gamma\left(\frac{\alpha_n+\alpha_m}{2} + n + m - k - j \right) \Gamma\left(\frac{\beta_n+\beta_m}{2} + k + j \right). \end{aligned} \tag{45}$$

The orthogonality of the wavefunctions can be proven using the Hermitian analog of (23), and the normalization constants can also be determined from a formula that is almost identical to (25):

$$J_{nn} = |C_n|^2 2^{\alpha_n+\beta_n-1} \frac{\Gamma(\alpha_n+n+1)\Gamma(\beta_n+n+1)}{n!\Gamma(\alpha_n+\beta_n+n+1)} \left(\frac{1}{\alpha_n} + \frac{1}{\beta_n} \right) \tag{46}$$

$$= |C_n|^2 2^{2s-2n} \frac{\Gamma(s+1+\lambda/(s-n))\Gamma(s+1-\lambda/(s-n))}{n!\Gamma(2s-n+1)} \frac{s-n}{(s-n)^2 - \lambda^2/(s-n)^2}. \tag{47}$$

From (47) it is obvious that due to $E_n < 0$ and the regularity conditions (which set the argument of each gamma function to a positive value) J_{nn} is positive. The actual value of the normalization constant is

$$C_n = 2^{n-s}\left(\frac{n!\Gamma(2s-n+1)[(s-n)^2-\lambda^2/(s-n)^2]}{\Gamma(s+1+\lambda/(s-n))\Gamma(s+1-\lambda/(s-n))(s-n)}\right)^{1/2}. \qquad (48)$$

This agrees with the results of [36] if the substitutions $U_0 = s(s+1)$, $B_0 = 2\lambda, \alpha = 1$ are made and the units are chosen as $\hbar = 2m = 1$. It has to be mentioned that there is a misprint in equation (3.7) of [36]: the correct formula should contain $-B_0$ instead of U_0.

4.1. Scattering states

The mathematical formulation for the real Rosen–Morse II potential is rather similar to that of the $\mathcal{PT}$-symmetric one; however, the physical interpretation of the results differs in several respects. The evaluation of the reflection and transmission coefficients can be done in the same way as in the $\mathcal{PT}$-symmetric case with the difference that the wave numbers k and k' are now real and can both be chosen positive. This means replacing k' with $-k'$ in equations (38) and (39). These equations reproduce the results of [37] for $E > |V(\pm\infty)| = 2|\Lambda|$.

Changing the direction of the incoming wave also results in expressions obtained from (40) and (41) by replacing k' by $-k'$. However, since k and k' are real, making use of the relation $\Gamma(z^*) = [\Gamma(z)]^*$ it can be seen that the reflection coefficient can change only up to a phase factor, while the transmission coefficient changes by a real factor.

For $|E| < 2|\Lambda|$ the scattering solution will decay exponentially in one direction, so there will be only a reflected wave in that case.

5. Discussion

This section aims to show the detailed comparison of the bound-state spectra, the normalization constants of the corresponding eigenfunctions as well as scattering aspects in the case of the $\mathcal{PT}$-symmetric and real Rosen–Morse II potentials. In this respect, the main features can be summarized as follows.

(i) There exist in both potentials at most a finite number of bound states, the maximum value of the principal quantum number being limited by the inequalities (20) and (44), respectively. All the bound-state energies are real in both cases.

(ii) In general, the number of bound states in the real potential is smaller than that in its complex counterpart. Moreover, as a consequence of equation (44), the real potential cannot support bound states at all when $s \leqslant |\lambda|^{1/2}$. In the complex potential, however, there always exists at least one bound state which is the ground state corresponding to $n = 0$. The energy of this universal ground state is $E_0 = -s^2 + \lambda^2/s^2$.

It is notable that the existence of a local minimum of the real Rosen–Morse II potential, which does actually occur whenever $s(s+1) > |\lambda|$, is a necessary but not a sufficient condition for the existence of at least one bound state in this potential. As mentioned above, the sufficient condition is $s^2 > |\lambda|$, which requires that the potential well be 'deep enough'. As is well known, this circumstance is connected to Heisenberg's uncertainty relations. A bounded particle is always localized to some extent in the potential well and thus, in its momentum and kinetic energy an indeterminacy occurs. For this reason, the non-symmetric potential wells must be sufficiently deep in order to be able to support bound states. Surprisingly, in the case of the $\mathcal{PT}$-symmetric Rosen–Morse II potential this well-known fact of the classical quantum mechanics does not hold. In this potential, at least one bound state always exists. This feature might be due to the imaginary potential component, which in contrast to the real component is asymptotically non-vanishing, and thus plays a more significant role in this case than in other potentials.

(iii) In the complex Rosen–Morse II and the real and symmetric Pöschl–Teller potential, which is the $\Lambda = 0$ special case of (2), the number of bound states is always the *same*, regardless the value of λ, i.e.

$$n_{\max}(\mathcal{PT}\ \text{RM II}) = n_{\max}(\text{Poschl–Teller}). \tag{49}$$

(iv) In the three potentials mentioned up to now the energy levels corresponding to the same value of n, when exist, are shifted upward with respect to each other as follows:

$$E_n(\text{real RM II}) < E_n(\text{Poschl–Teller}) < E_n(\mathcal{PT}\ \text{RM II}). \tag{50}$$

(v) While in the real Rosen–Morse II and the Pöschl–Teller potential the bound-state energies are always negative, in the $\mathcal{PT}$-symmetric complex Rosen–Morse II potential the bound states of non-negative energy can exist whenever the condition

$$s - |\lambda|^{1/2} \leqslant n < s \tag{51}$$

is met. The equality case $n = s - |\lambda|^{1/2}$ of (51) yields a zero-energy bound state. Thus, the energy of the universal ground state in the $\mathcal{PT}$-symmetric Rosen–Morse II potential is zero when $|\lambda| = s^2$, in agreement with the point (ii). The energies of all the exited states are positive in this case. Moreover, when $|\lambda| > s^2$ the whole bound-state spectrum of the $\mathcal{PT}$-symmetric Rosen–Morse II potential is positive. When $|\lambda| > (1 + 5^{1/2})s^2/2$, the whole bound-state spectrum is shifted even above the largest asymptotic value $\lim_{x\to\pm\infty} |V(x)| = |\lambda|$ of the complex potential.

Bearing in mind that in the $\mathcal{PT}$-symmetric Rosen–Morse II potential all the bound-state energies are positive for $|\lambda| > s^2$, it is reasonable to assume that this phenomenon might be due to the dominating non-Hermitian potential component, which, contrary to the real (Pöschl–Teller) potential component, does not vanish asymptotically. We note that no similar behavior was observed for the $\mathcal{PT}$-symmetric Scarf II potential [19] which has the same real component, but its imaginary component vanishes asymptotically. In that case, increasing non-Hermiticity led to the spontaneous breakdown of $\mathcal{PT}$-symmetry. We mention that the occurrence of the positive energy bound states in the $\mathcal{PT}$-symmetric Rosen–Morse II potential has already been pointed out by Znojil [9]. There a parallel has been drawn with the empirically observed positive energy eigenvalues in the $V(x) = \mathrm{i}x^3$ potential [1] which is also purely imaginary asymptotically. (The real and positive nature of the spectrum of this latter potential and some of its generalization has been proven later analytically [38, 39].) It appears that for the occurrence of the positive energy solutions an asymptotically non-vanishing, but finite imaginary potential component is also sufficient. Nevertheless, it is not immediately obvious why the whole positive bound-state energy spectrum can be shifted arbitrarily far above the largest asymptotic value $\lim_{x\to\pm\infty} |V(x)| = |\lambda|$ of the $\mathcal{PT}$-symmetric Rosen–Morse II potential.

(vi) While in the real Rosen–Morse II potential, the bound state eigenfunctions can always be normalized to +1, the pseudo-norm of the bound-state eigenfunctions in the complex Rosen–Morse II potential follows the $(-1)^n$ rule.

The sign change of the pseudo-norm according to the $(-1)^n$ rule has also been observed for the $\mathcal{PT}$-symmetric Scarf I [24] and Rosen–Morse I [25] potentials, as well as for other $\mathcal{PT}$-symmetric potentials with an infinite number of bound states [33, 34], but it differs from the case of the $\mathcal{PT}$-symmetric Scarf II potential [19], which also has a finite number of bound states.

(vii) The results obtained for the transmission and reflection coefficients of the $\mathcal{PT}$-symmetric and real Rosen–Morse potentials are similar from the mathematical point of view, but their physical content differs significantly. Due to the asymmetric structure of the potential, the asymptotic wave numbers differ in both cases for $x \to \infty$ and $x \to -\infty$. A major difference is, however, that the wave numbers are complex for the $\mathcal{PT}$-symmetric potential. Changing the direction of the incoming wave changes the transmission and reflection coefficients in both cases. In the case of the real potential, the transmission coefficient changes with a real factor, while the reflection coefficient picks up only an extra phase. In the $\mathcal{PT}$-symmetric case, the transmission coefficient changes by a complex phase factor, while the reflection coefficient changes more drastically, corresponding to the handedness effect observed for local $\mathcal{PT}$-symmetric scattering potentials in general.

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